A New Method for Ranking Exponential Fuzzy Numbers with use Weighted Average and Weighted Width in TRD Distance

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ABSTRACT
In this paper, we want to present a new method for ranking exponential fuzzy numbers with use weighted average and weighted width in TRD distance. Weighted average and weighted width of exponential fuzzy numbers combines in TRD distance for ranking method. A simpler and easier approach is proposed for the ranking of generalized trapezoidal fuzzy numbers. For the validation the results of the proposed approach are compared with different existing approaches.

Keywords: Exponential fuzzy numbers, TRD distance, weighted average, weighted width.

1. Introduction
Ranking fuzzy numbers are an important tool in decision making. In fuzzy decision analysis, fuzzy quantities are used to describe the performance of alternatives in modeling a real-world problem. Various ranking procedures have been developed since 1976 when the theory of fuzzy sets were first introduced by Zadeh [27]. Ranking fuzzy numbers were first proposed by Jain [13] for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Bortolan and Degani [4] reviewed some of these ranking methods [3,13,18,19] for ranking fuzzy subsets. Chen [6] presented ranking fuzzy numbers with maximizing set and minimizing set. Dubois and Prade [12] presented the mean value of a fuzzy number. Lee and Li [16] presented a comparison of fuzzy numbers based on the probability measure of fuzzy events. Delgado et al. [11] presented a procedure for ranking fuzzy numbers. Campos and Munoz [5] presented a subjective approach for ranking fuzzy numbers. Kim and Park [14] presented a method of ranking fuzzy numbers with index of optimism. [17] and Wang and Lee [25] also used the centroid concept in developing their ranking index. Chen and Chen [7] presented a method for ranking generalized trapezoidal fuzzy numbers. Abbasbandy and Hajjari [1] introduced a new approach for ranking of trapezoidal fuzzy numbers based on the left and right spreads at some $\alpha$-levels of trapezoidal fuzzy numbers. Chen and Chen [8] presented a method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. Also, Allahviranloo [2] proposed a
method for ranking of fuzzy numbers using new weighted distance. Rezvani ([18],[23]) evaluated the system of ranking fuzzy numbers. Moreover, Rezvani [22] proposed a new method for ranking in areas of two generalized trapezoidal fuzzy numbers.

In this paper, we want to present a new method for ranking exponential fuzzy numbers with use weighted average and weighted width in TRD distance. Weighted average and weighted width of exponential fuzzy numbers combines in TRD distance for ranking method. A simpler and easier approach is proposed for the ranking of generalized trapezoidal fuzzy numbers. For the validation the results of the proposed approach are compared with different existing approaches.

2. Preliminaries

Definition 1. Generally, a generalized fuzzy number $A$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_A$ satisfies the following conditions,

i) $\mu_A$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0,1]$,

ii) $\mu_A(x) = 0$, $-\infty < x \leq c$,

iii) $\mu_A(x) = L(x)$ is strictly increasing on $[c,a]$,

iv) $\mu_A(x) = w$, $a \leq x \leq b$,

v) $\mu_A(x) = R(x)$ is strictly increasing on $[b,d]$,

vi) $\mu_A(x) = 0$, $d \leq x < \infty$

where $0 < w \leq 1$ and $a,b,c$ and $d$ are real numbers. We call this type of generalized fuzzy number a trapezoidal fuzzy number, and it is denoted by $A = (c,a,b,d;w)_{LR}$.

When $w = 1$, this type of generalized fuzzy number is called normal fuzzy number and is represented by $A = (c,a,b,d)_{LR}$.

However, these fuzzy numbers always have a fix range as $[c, d]$ . Here, we define its general form as follows:

$$f_A(x) = \begin{cases} \frac{e^{-[(a-x)/\gamma]} - x}{w} & \text{if } x \leq a \\ \frac{e^{-[(a-b)/\beta]} - a}{b} & \text{if } a \leq x \leq b \\ \frac{e^{-[(a-b)/\beta]} - b}{x} & \text{if } b \leq x \end{cases} \quad (1)$$

where $0 < w \leq 1$ and $a, b$ are real numbers and $\gamma, \beta$ are positive real numbers. We denote this type of generalized exponential fuzzy number as $A = (a,b,\gamma,\beta;w)_E$ . Especially, when $w = 1$, we denote it as $A = (a,b,\gamma,\beta)_E$.

We define the representation of generalized exponential fuzzy number based on the integral value of graded mean h-level as follow. Let the generalized exponential fuzzy number $A = (a,b,\gamma,\beta;w)_E$, where $0 < w \leq 1$ and $a, b$ are real numbers formula (1).

Now, let two monotonic functions be

$$L(x) = \frac{e^{-[(a-x)/\gamma]}}, \quad R(x) = \frac{e^{-[(a-b)/\beta]}}{x} \quad (2)$$

Then the inverse functions of function $L$ and $R$ are $L^{-1}$ and $R^{-1}$ respectively.
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\[ L^{-1}(h) = a - \gamma (\ln \frac{w}{h}), \] (3)

\[ L^{-1}(h) = b + \beta (\ln \frac{w}{h}). \] (4)

**Definition 2.** The following values constitute the weighted averaged representative and weighted width, respectively, of the fuzzy number \( A \)

\[ I(A) = \int_{0}^{1}[CL_{A}(\alpha)+(1-C)R_{A}(\alpha)]d\alpha \] (5)

and\[ D(A) = \int_{0}^{1}[R_{A}(\alpha)-L_{A}(\alpha)]f(\alpha)d\alpha \] (6)

Here \( 0 \leq C \leq 1 \) denotes an "optimism/pessimism" coefficient in conducting operations on fuzzy numbers. The function \( f(\alpha) \) is nonnegative and increasing function on \([0, 1]\) with \( f(0) = 0, f(1) = 1 \) and \( \int_{0}^{1}f(\alpha)d\alpha = \frac{1}{2} \). The function \( f(\alpha) \) is also called weighting function. In actual applications, function \( f(\alpha) \) can be chosen according to the actual situation. In this article, in practical case, we assume that \( f(\alpha) = \alpha \).

3. Usage TRD Distance in Ranking Exponential Fuzzy Numbers

In this section some important results, that are useful for the proposed approach, are proved.

**Theorem 1.** The values constitute of weighted averaged representative and weighted width, of the exponential fuzzy number \( A \) are following

\[ I(A) = w[C\gamma(e^{-\frac{\alpha}{\gamma}} - e^{-\frac{1}{\gamma}})] + w[(1-C)\beta(e^{\frac{b}{\beta}} - e^{-\frac{b-1}{\beta}})] \] (7)

and\[ D(A) = w[\beta^2 e^{\frac{b}{\beta}} - e^{\frac{b-1}{\beta}} (\beta + \beta^2)] - w[\gamma^2 e^{-\frac{1}{\gamma}} + e^{-\frac{\alpha}{\gamma}} (\gamma - \gamma^2)] \] (8)

**Proof:**

\[ I(A) = \int_{0}^{1}[CL_{A}(\alpha)+(1-C)R_{A}(\alpha)]d\alpha = \int_{0}^{1}[Cwe^{-(a-\alpha)/\gamma} + (1-C)we^{-(a-b)/\beta}]d\alpha \]

\[ = C\gamma[e^{-\frac{1-a}{\gamma}} - e^{-\frac{a}{\gamma}}] + (1-C)w\beta[e^{\frac{b}{\beta}} - e^{-\frac{b-1}{\beta}}] = C\gamma[e^{-\frac{1-a}{\gamma}} - e^{-\frac{a}{\gamma}}] + \\
\text{and} \quad w[(1-C)\beta(e^{\frac{b}{\beta}} - e^{-\frac{b-1}{\beta}})] \]

\[ D(A) = \int_{0}^{1}[R_{A}(\alpha)-L_{A}(\alpha)]f(\alpha)d\alpha = \int_{0}^{1}[we^{-(a-b)/\beta} - we^{-(a-x)/\gamma}]f(\alpha)d\alpha \]
Theorem 2. TRD distance between the exponential fuzzy number \( A \) as following

\[
\text{TRD}(A) = \sqrt{(I(A))^2 + (D(A))^2}
\]

(9)

Theorem 3. TRD distance between the exponential fuzzy numbers \( A \) and \( B \) as following

\[
\text{TRD}(A, B) = \sqrt{(I(A) - I(B))^2 + (D(A) - D(B))^2}
\]

(10)

Definition 3. Let \( A_1 = (a_1, b_1, \alpha_1, \beta_1; w_1)_E \) and \( A_2 = (a_2, b_2, \alpha_2, \beta_2; w_2)_E \) be two generalized exponential fuzzy numbers, then

i) If \( \text{TRD}(A_1) < \text{TRD}(A_2) \), then \( A_1 < A_2 \),

ii) If \( \text{TRD}(A_1) > \text{TRD}(A_2) \), then \( A_1 > A_2 \),

iii) If \( \text{TRD}(A_1) = \text{TRD}(A_2) \), then \( A_1 \sim A_2 \).

3.1. Method to Find the Values of TRD(A) and TRD(B)

Let \( A_1 = (a_1, b_1, \alpha_1, \beta_1; w_1)_E \) and \( A_2 = (a_2, b_2, \alpha_2, \beta_2; w_2)_E \) be two generalized exponential fuzzy numbers, then use the following steps to find the values of \( \text{TRD}(A_1) \) and \( \text{TRD}(A_2) \)

* Step 1
  Find \( I(A_1) \) and \( I(A_2) \)

* Step 2
  Find \( D(A_1) \) and \( D(A_2) \)

* Step 3
  Calculation \( \text{TRD}(A_1) \) and \( \text{TRD}(A_2) \) and use definition 3. , for ranking this method.

4. Validation of another Proposed Method

Example 1. Let \( A = (0.2, 0.4, 0.6, 0.8; 0.35) \) and \( B = (0.1, 0.2, 0.3, 0.4; 7) \) be two generalized trapezoidal fuzzy number, then

* Step 1
  \( I(A) = 0.31C + 0.33, \quad I(B) = 3.6C + 0.42 \)

* Step 2
  \( D(A) = -0.14, \quad D(B) = -2.8 \)
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* Step 3

\[ TRD(A) = \sqrt{(0.31C + 0.33)^2 + (-0.14)^2} \]

For

\[ C = 0 \Rightarrow TRD(A) = 0.36, \]
\[ C = 0.5 \Rightarrow TRD(A) = 0.5, \]
\[ C = 1 \Rightarrow TRD(A) = 0.65, \]

\[ TRD(B) = \sqrt{(3.6C + 0.42)^2 + (-2.8)^2} \]

For

\[ C = 0 \Rightarrow TRD(B) = 2.83, \]
\[ C = 0.5 \Rightarrow TRD(B) = 3.57, \]
\[ C = 1 \Rightarrow TRD(B) = 4.9, \]

Then, \( TRD(A) < TRD(B) \Rightarrow A < B \).

**Example 2.** Let \( A = (0.1, 0.2, 0.4, 0.5; 1) \) and \( B = (0.1, 0.3, 0.3, 0.5; 1) \) be two generalized trapezoidal fuzzy number, then

* Step 1

\[ I(A) = 2.8C + 0.65, \quad I(B) = 4.91C + 0.84 \]

* Step 2

\[ D(A) = -1.03, \quad D(B) = -3.98 \]

* Step 3

\[ TRD(A) = \sqrt{(2.8C + 0.65)^2 + (-1.03)^2} \]

For

\[ C = 0 \Rightarrow TRD(A) = 1.22, \]
\[ C = 0.5 \Rightarrow TRD(A) = 2.29, \]
\[ C = 1 \Rightarrow TRD(A) = 3.6, \]

\[ TRD(B) = \sqrt{(4.91C + 0.84)^2 + (-3.98)^2} \]

For

\[ C = 0 \Rightarrow TRD(B) = 4.07, \]
\[ C = 0.5 \Rightarrow TRD(B) = 5.17, \]
\[ C = 1 \Rightarrow TRD(B) = 6.99, \]

Then, \( TRD(A) < TRD(B) \Rightarrow A < B \).

**Example 3.** Let \( A = (0.1, 0.2, 0.4, 0.5; 1) \) and \( B = (1, 1, 1, 1; 1) \) be two generalized trapezoidal fuzzy number, then

* Step 1

\[ I(A) = 2.8C + 0.65, \quad I(B) = -1.08C + 1.71 \]

* Step 2

\[ D(A) = -1.03, \quad D(B) = -0.34 \]
Step 3

\[ TRD(A) = \sqrt{[2.8C + 0.65]^2 + [-1.03]^2} \]

For

\[ C = 0 \Rightarrow TRD(A) = 1.22, \]
\[ C = 0.5 \Rightarrow TRD(A) = 2.29, \]
\[ C = 1 \Rightarrow TRD(A) = 3.6, \]

\[ TRD(B) = \sqrt{[-1.08C + 1.71]^2 + [0.34]^2} \]

For

\[ C = 0 \Rightarrow TRD(B) = 1.14, \]
\[ C = 0.5 \Rightarrow TRD(B) = 1.22, \]
\[ C = 1 \Rightarrow TRD(B) = 0.72, \]

Then, \( TRD(A) > TRD(B) \Rightarrow A > B \).

**Example 4.** Let \( A = (-0.5, -0.3, -0.3, -0.1; 1) \) and \( B = (0.1, 0.3, 0.3, 0.5; 1) \) be two generalized trapezoidal fuzzy number, then

* Step 1

\[ I(A) = -42518.8C + 42518.83, \quad I(B) = 4.91C + 0.84 \]

* Step 2

\[ D(A) = 38268.9, \quad D(B) = -3.98 \]

* Step 3

\[ TRD(A) = \sqrt{[-42518.8C + 42518.83]^2 + [38268.9]^2} \]

For

\[ C = 0 \Rightarrow TRD(A) = 57204.5, \]
\[ C = 0.5 \Rightarrow TRD(A) = 43777.5, \]
\[ C = 1 \Rightarrow TRD(A) = 38268.9, \]

\[ TRD(B) = \sqrt{[4.91C + 0.84]^2 + [-3.98]^2} \]

For

\[ C = 0 \Rightarrow TRD(B) = 4.07, \]
\[ C = 0.5 \Rightarrow TRD(B) = 5.17, \]
\[ C = 1 \Rightarrow TRD(B) = 6.99, \]

Then, \( TRD(A) > TRD(B) \Rightarrow A > B \).

**Example 5.** Let \( A = (0.3, 0.5, 0.5, 1; 1) \) and \( B = (0.1, 0.6, 0.6, 0.8; 1) \) be two generalized trapezoidal fuzzy number, then

* Step 1

\[ I(A) = 0.7C + 1.04, \quad I(B) = 0.97C + 1.2 \]

* Step 2

\[ D(A) = -0.71, \quad D(B) = -0.89 \]
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* Step 3

\[ TRD(A) = \sqrt{[0.7C + 1.04]^2 + [-0.71]^2} \]

For
\[ C = 0 \Rightarrow TRD(A) = 1.28, \]
\[ C = 0.5 \Rightarrow TRD(A) = 1.58, \]
\[ C = 1 \Rightarrow TRD(A) = 1.89, \]
\[ TRD(B) = \sqrt{[0.97C + 1.2]^2 + [-0.89]^2} \]

For
\[ C = 0 \Rightarrow TRD(B) = 1.49, \]
\[ C = 0.5 \Rightarrow TRD(B) = 1.9, \]
\[ C = 1 \Rightarrow TRD(B) = 2.34, \]
Then, \( TRD(A) < TRD(B) \Rightarrow A < B \).

**Example 6.** Let \( A = (0, 0.4, 0.6, 0.8; 1) \) and \( B = (0.2, 0.5, 0.5, 0.9; 1) \) and \( C = (0.1, 0.6, 0.7, 0.8; 1) \) be two generalized trapezoidal fuzzy number, then

* Step 1

\[ I(A) = 1.63C + 0.94, \quad I(B) = 1.1C + 1.03, \quad I(C) = 0.7C + 1.2 \]

* Step 2

\[ D(A) = -1.3, \quad D(B) = -1.11, D(C) = -0.7 \]

* Step 3

\[ TRD(A) = \sqrt{[1.63C + 0.94]^2 + [-1.3]^2} \]

For
\[ C = 0 \Rightarrow TRD(A) = 1.6, \]
\[ C = 0.5 \Rightarrow TRD(A) = 2.18, \]
\[ C = 1 \Rightarrow TRD(A) = 2.88, \]
\[ TRD(B) = \sqrt{[1.1C + 1.03]^2 + [-1.11]^2} \]

For
\[ C = 0 \Rightarrow TRD(B) = 1.51, \]
\[ C = 0.5 \Rightarrow TRD(B) = 1.93, \]
\[ C = 1 \Rightarrow TRD(B) = 2.4, \]
\[ TRD(C) = \sqrt{[0.7C + 1.2]^2 + [-0.7]^2} \]

For
\[ C = 0 \Rightarrow TRD(c) = 1.39, \]
\[ C = 0.5 \Rightarrow TRD(c) = 1.7, \]
\[ C = 1 \Rightarrow TRD(c) = 2.02, \]
Then, \( TRD(A) > TRD(B) > TRD(C) \Rightarrow A > B > C \).
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**Example 7.** Let $A = (0.1, 0.2, 0.4, 0.5; 1)$ and $B = (-2, 0, 2; 1)$ be two generalized trapezoidal fuzzy number, then

* Step 1

$I(A) = 2.18C + 0.65$, $I(B) = -0.78C + 0.78$

* Step 2

$D(A) = -1.03$, $D(B) = 0.36$

* Step 3

$$TRD(A) = \sqrt{[2.81C + 0.65]^2 + [-1.03]^2}$$

For

$C = 0 \Rightarrow TRD(A) = 1.22,$

$C = 0.5 \Rightarrow TRD(A) = 1.58,$

$C = 1 \Rightarrow TRD(A) = 3.61,$

$$TRD(B) = \sqrt{[-0.78C + 0.78]^2 + [0.36]^2}$$

For

$C = 0 \Rightarrow TRD(B) = 0.74,$

$C = 0.5 \Rightarrow TRD(B) = 0.53,$

$C = 1 \Rightarrow TRD(B) = 0.36$

Then, $TRD(A) > TRD(B) \Rightarrow A > B$.

For the validation of the proposed ranking function, in Table 1, it is shown that this approach is very simple and easy to apply in the real life problems.

<table>
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<td>A&gt;B</td>
<td>A&gt;B</td>
<td>A&gt;B</td>
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**Table 1.** A comparison of the ranking results for different approaches

5. **Conclusion**

Weighted average and weighted width of exponential fuzzy numbers combines in TRD distance for ranking method. A simpler and easier approach is proposed for the ranking of generalized trapezoidal fuzzy numbers.
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